The Big Lockdown Math Off

Cleaving Cats

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March 30, 2020

We all like to talk about randomness and know how hard it is to recreate and gauge when things are actually random. I think that randomness is fascinating and a Geometer, the mixture of the two creates a favourite bit of Maths and an enjoyable coding task.

I want the gif here, it won't come into the LaTeX, it is called output.gif

1 Tori and Transfomations

A Torus is also known as a Donut, it is a continuous surface that is the same as the edible snack. The surface can also be covered using rectangular strips of a certain length and width, but also, less intuitively be covered using triangles, with caveats. With the triangles, these can be made by skewing squares and rectangles, this can be done using linear transformations and linear systems of equations.

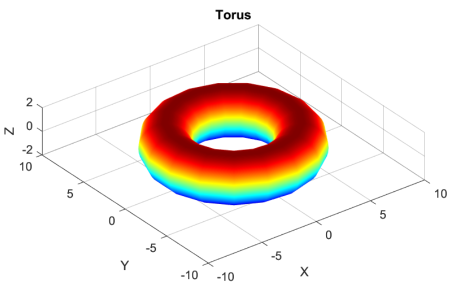

Figure 1: A Torus

In relation to the matrices, we can define a shear in parallel to the x axis: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and in the y axis: $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$

We can define a linear tranformation using a matrix and a vector, $v\mathbf{A} = v'$ where $v = (x \quad y)^T$. So we could take a transformation parallel to the x axis and parallel to the y axis one after another. This would then produce rectangles which would nicely cover the torus.

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 + a^2 & a \\ a & 1 \end{pmatrix}$$

If we just take the generic and simplest non-trivial case, of $n = 1$, we end up at a particular matrix. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, this matrix is equivalent to the mapping: $\Gamma(x, y) \to (2x + y,\, x + y)$. Remember the caveat I mentioned earlier, this caveat is the fact you could get overlapping triangles wrapping around the torus parallel to the major radii. To stop this, you need to place a thing called a modulo N operator, this will wrap any number greater than N to the amount it is greater than N, i.e. $35 \mod 6 = 5$. This makes the equations now: $\Gamma(x, y) \to (2x + y,\, x + y) \mod N$, but N is a number and one we must choose carefully.

2 Kittens of Chaos

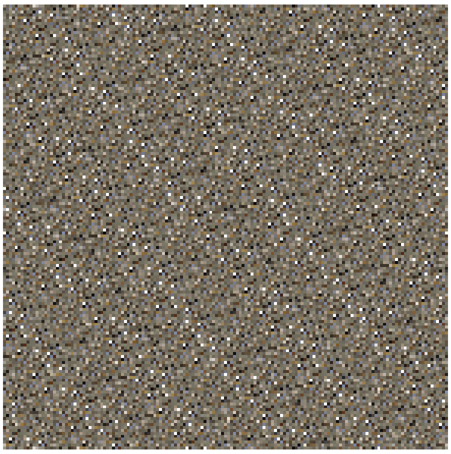

Figure 2: cat_006

At this point, we must go into greater detail about what we are deriving here. This is a system that you would expect to output chaos given what it does to an image, let me show you a frame from a couple of iterations in. Figure 2 shows almost white noise that you would expect to see on an untuned analogue television, this came from a perfectly normal image of a cat. To deal with pixels individually, we will let N=1.[1]. This is an invertible process, so we can get to this state, and we could go backwards by taking an inverse.

We know many cats are little fluffy gremlins, but this takes it to the extreme, this is Chaos. For something to be chaotic, it must tick the following three conditions:

- Sensitive to Initial Conditions; in this mapping, the initial conditions are the images you input
- It is to be topologically transitive; the outputs of the mapping gets flung out so much that there isn't any overlapping of the outputs
- The mapping has dense periodic orbits; every point is either arbitrary close to a periodic point or is a periodic point.

If we input two different images, then the output would be different colours and would produce drastically different outputs. Even if we just edited all the white in Figure 2 to

[1]The more astute of you may notice that there's a problem here, what about integers, this would place many pixels into $(0, 0)$, we define a quotient space $\mathbb{T}^2$, which is $\mathbb{R}^2 \setminus \mathbb{Z}^2$. Now $\Gamma : \mathbb{T}^2 \to \mathbb{T}^2$.

be yellow, it would be drastically different.

Take a small area of pixels in a border where the image is being split into triangles. The set is to be divided in half, and they are to be placed away from each other. Then they may be cut again and again and again, producing an almost dissolving effect. So the original set is mixed into the image. This is why topological transitivity is called a mixing effect. As seen in Figure 3, the triangles/quadrilaterals are picked between the subset and then they are split, and so the set is mixed into the set.

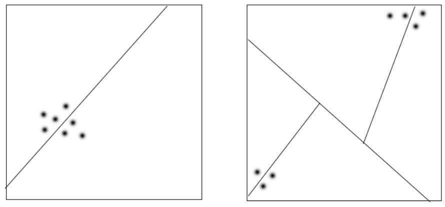

Figure 3: The Mixing Effect

To show that there are dense periodic points, it suffices to show that all points are periodic, to do this, we shall look at element cat_000 and cat_120:

(a) cat_000

(b) cat_120

Figure 4: Showing periodic points

Just a minute... thats the same image... this chaotic sequence is periodic. It does all these weird and wonderful shears on the torus and eventually it just turns back into the original state of the torus!

This now proves that the mapping is chaotic mapping, which is so aptly named the Arnold Cat Map.

3 Flows of Milk

In dynamical systems like this, to see geometric flow we study the eigenvalues of the associated matrix of the system:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda) - 1 \\ &= \lambda^2 - 3\lambda + 2 - 1 \\ &= \lambda^2 - 3\lambda + 1 \\ \implies \lambda &= \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

$\therefore \lambda_1, \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$

In systems of the form of differential equations, we could take information just from this. Still, we are working with recurrence relations, so we need to consider the Liapunov Exponents. These exponents are just the eigenvalues with the numerator natural logarithmed: $\Lambda_{1,2} = \frac{\ln(3 \pm \sqrt{5})}{2}$. We are allowed to now use the same rules as eigenvalues and differential equations. So we know it must be a saddle point, which is of the form of Figure 5.

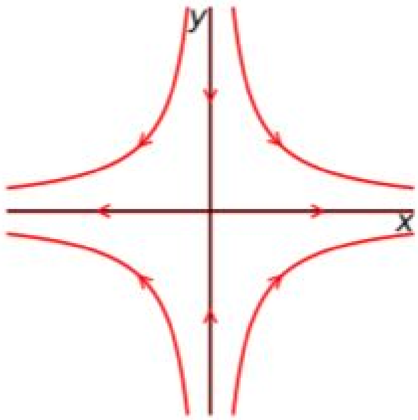

Figure 5: A sketch of the form

Given the output of the mapping, we know that the point at (0, 0) must be static and then the curve in the quadrants must be cyclic. Doing some sketching and some exploration of eigenvectors, you can approach the diagram in Figure 6.

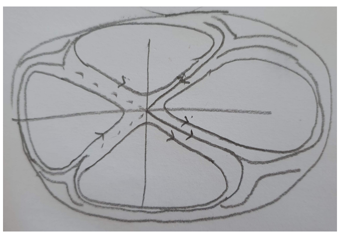

Figure 6: A Guess of the flows given the one equilibria

4 Why is this brilliant?

The Arnold Cat Map is an algorithm that jumbles up an image to near Chaos and ends up with the picture back in the original position. This is something you'd never expect from it at first glance. This is something of great interest and is the main reason I love Mathematics, it's all about delicately pulling away the other less colourful petals to get to the beautiful rose inside.

I created some python to create the gif, it is here:

```
from PIL import Image
import numpy as np
import glob

# load image
im = np.array(Image.open("input.jpg"))
N = im.shape[0]

# create x and y components of Arnold's cat mapping
x,y = np.meshgrid(range(N+1),range(N+1))
xmap = (2*x+y) % N
ymap = (x+y) % N

for i in range(N+1):
 result = Image.fromarray(im)
 result.save("./images/cat_%03d.png" % i)
 im = im[xmap,ymap]
 print(i)

fp_in = "./images/cat_*.png"
fp_out = "./output.gif"

# https://pillow.readthedocs.io/en/stable/handbook/image-file-formats.html#gif
img, *imgs = [Image.open(f) for f in sorted(glob.glob(fp_in))]
img.save(fp=fp_out, format='GIF', append_images=imgs, save_all=True, duration=120, loop=1)
```

Code from ./figures/catMap.py

output.gif again please